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1995 J. Phys. A: Math. Gen. 28 5331

(http://iopscience.iop.org/0305-4470/28/18/022)

J. Phys. A: Math. Gen. 28 (1995) 5331–5347. Printed in the UK

Quantum mechanics of multi-prong potentials

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Received 6 March 1995

Abstract. Some aspects of the bound state and scattering properties of a quantum mechanical particle in an arbitrary scalar N-prong potential are considered. Such a study is relevant in applications to mesoscopic devices. Multi-prong potentials are in some way intermediate between one- and two-dimensional systems. In contrast to the one-dimensional situation, where there is no degeneracy, the energy levels for the case of N identical prongs exhibit an alternating pattern of non-degeneracy and $(N-1)$-fold degeneracy. We generalize the techniques of supersymmetric quantum mechanics to multi-prong systems and generate solutions to new N-prong systems. Solutions for prongs of arbitrary lengths are developed. Scattering on piecewise constant potentials and tunnelling in N-well potentials are discussed in detail. Since our treatment is for general values of N, the results can be studied in the large N-limit. A somewhat surprising result is that a free particle incident on an N-prong vertex undergoes continuously increased backscattering as the number of prongs is increased.

1. Introduction

Various aspects of the solutions of the Schrödinger equation for both scalar and vector potentials on a wide variety of networks have been discussed by several authors [1–15]. Recent investigations have been motivated in part by the considerable interest in mesoscopic systems and the experimental observation of persistent currents [16]. These studies cover the situations of constant scalar potentials, identical on all prongs, vector potentials associated with uniform magnetic fields, or δ-function potentials at the network vertices.

In this paper, we discuss the bound state and scattering properties of a single quantum mechnical particle moving in an arbitrary scalar N-prong potential. Our treatment clearly shows the generalization from the two-prong case, which is just the familiar one-dimensional (particle on a line) problem. For simplicity, here we do not consider δ-function potentials at the vertex. Properties, such as degeneracies of eigenstates, boundary conditions, scattering and tunnelling on such N-prong systems are not *a priori* evident, since these systems are in some ways intermediate between one and two dimensions. For example, in contrast to the one-dimensional situation where there is no degeneracy, the energy levels for the case of N identical prongs exhibit an alternating pattern of non-degeneracy and $(N-1)$-fold degeneracy. The concept of supersymmetry has yielded many interesting results for one-dimensional quantum mechanics [17, 18]. We find that that these techniques can be generalized to N-prong systems and generate solutions to new potentials. Solutions for prongs of arbitrary lengths are developed in detail. In the case of identical prongs, the eigenfunctions of the original potential and its supersymmetric N-prong partner potential are degenerate. Scattering on piecewise constant potentials and tunnelling in N-well potentials

0305-4470/95/185331+17$19.50

are discussed. Since our treatment is for general values of N, the results can be studied in the large-N limit. A somewhat surprising result is that a free particle incident on an N-prong vertex undergoes more backscattering as the number of prongs is increased. This leads to zero transmission and full reflection in the limit of infinitely many prongs, merely by the topology of the situation. While heuristic considerations would suggest that transmission into each single prong goes down as the number of competing prongs increases; this is not enough to explain why the reflection coefficient should go to one in this limit. We extend the familiar discussion of tunnelling in a double-well potential to the case of N-well potential systems. We find that the tunnelling of a wavepacket, localized in one prong, into other prongs decreases as the number of prongs is increased and goes to zero in the limit of infinite N; there always remains a remnant of the wavepacket in the original well.

The plan of this paper is as follows. In section 2, we define N-prong potentials and give the boundary conditions which the wavefunction must satisfy. The normalization of wavefunctions and their orthogonality properties are also discussed. Section 3 contains a discussion of bound states for potentials with N identical prongs. The general solution is derived and is shown to be closely related to the solutions of a symmetric one-dimensional potential. Section 4 contains a discussion of bound states in N-prong potentials with non-identical prongs. Several analytical and numerical solutions are given to illustrate properties of the eigenfunctions. In particular, we find that a generalized version of the usual rule about one extra node for each higher eigenstate is probably true but the theorem about non-degeneracy, which holds in one-dimensional situations, is not. Generalization of supersymmetry to N-prong systems and applications are discussed in section 5. Section 6 illustrates the use of lowest-order perturbation theory to obtain bound-state energies. Scattering is discussed in section 7. For the special case of no potential in any of the N prongs, surprising results are obtained which are quite different from naive expectations. Finally, in section 8, we generalize the concept of tunnelling to the case of N-well potential systems.

2. Multi-prong potentials

We are addressing the problem of formulating and solving the Schrödinger equation for a single particle constrained to remain in a space made up of N lines meeting at a vertex point (figure 1). The prongs are labelled by the indices i ($i = 1, 2, \ldots, N$) and the position on any prong is given by the positive coordinate x_i, with $x_i = 0$ being the vertex point. The potential is fully specified by giving the scalar potentials $V_i(x_i)$ on each prong i. The overall wavefunction $\psi(\vec{x})$ is composed of the individual wavefunctions $\psi_i(x_i)$ on each prong i:

$$\psi(\vec{x}) \equiv \{\psi_1(x_1), \ldots, \psi_N(x_N)\}. \tag{1}$$

We are using the arrow notation to indicate the appropriate N-tuple. The physical requirement of single-valuedness of the wavefunction at the vertex implies for the component wavefunctions

$$\psi_1(0) = \psi_2(0) = \cdots = \psi_N(0). \tag{2}$$

In this paper, we will not be considering δ-function potentials at the vertex which have been discussed in [2, 6, 7]. Consequently, the second vertex condition requires the sum of all derivatives to add up to zero [1, 6]:

$$\sum_{i=1}^{N} \frac{\mathrm{d}\psi_i}{\mathrm{d}x_i}\bigg|_{x_i=0} = 0. \tag{3}$$

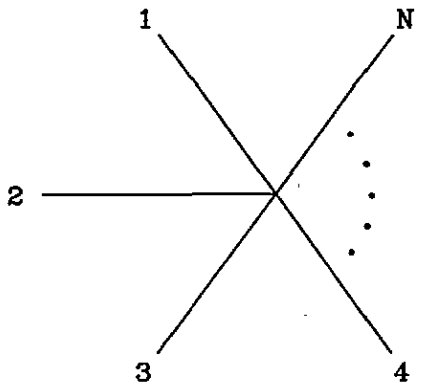

Figure 1. Schematic diagram of an N-prong potential. The position on any prong i is specified by the positive coordinate x_i, with $x_i = 0$ being the vertex. The angles between prongs play no role in any computations.

This can be readily derived as follows. Let us embed the multi-prong system in a three-dimensional space. Let τ be a neighbourhood that contains the vertex. For simplicity, assume that the system has all prongs in the x–y plane (polar angle $\theta = \frac{\pi}{2}$). Since the particle is never to be found between prongs, the wavefunction is defined to be zero there. In the neighbourhood τ, the wavefunction ψ will then be given by

$$\psi(\vec{r}) = \sum_{i=1}^{N} \psi(x_i)\delta(\theta - \pi/2)\,\delta(\phi - \phi_i)$$

where the ϕ_i are the azimuthal prong angles. The time-independent Schrödinger equation, reads $\nabla^2\psi = (V-E)\psi$, taking units in which $\hbar = 2m = 1$. We assume that the potential is finite in the above neighbourhood. Integrating the Schrödinger equation over a small sphere of radius ϵ within τ with the vertex as the centre, and using the divergence theorem (and then letting $\epsilon \to 0$), we get $\lim_{\epsilon\to 0}\int_A \nabla\psi\,\mathrm{d}A = 0$, which reduces to (3) for an N-prong potential. Note that for the special case $N = 2$, one has the standard one-dimensional situation and, taking into account the relative directions of x_1 and x_2, equation (3) is just the statement that the derivative $\mathrm{d}\psi/\mathrm{d}x$ is continuous across the vertex.

The above boundary conditions are not the most general possible. They are reasonable from physical considerations, and have been used in [9]. Other self-adjoint extentions have been considered by several authors [10, 11, 19]. Self-adjoint extensions are defined in [20, 21].

For the N-prong system, we call two functions $\psi(\vec{x})$ and $\phi(\vec{x})$ orthogonal if

$$\sum_{i=1}^{N}\int \psi_i^*(x_i)\phi_i(x_i)\,\mathrm{d}x_i = 0 \tag{4}$$

where the sum indicates that all prongs are included and each integral runs over the appropriate prong length. The normalization condition, which also includes all prongs, reads

$$\sum_{i=1}^{N}\int \psi_i^*(x_i)\psi_i(x_i)\,\mathrm{d}x_i = 1\,. \tag{5}$$

The eigenfunctions of the Schrödinger equation in a multi-prong potential not only have to satisfy the vertex conditions given in (2) and (3), but also appropriate boundary conditions at each prong end. These conditions depend on the energy E and the specific behaviour of

$V_i(x_i)$ for the maximum allowed value of x_i. Eigenfunctions corresponding to bound-state solutions have to vanish at the prong ends. For potentials which reach a constant value fast enough at the prong ends, one has standard plane wave solutions (e^{ikx_i}, e^{-ikx_i}) for the scattering states.

3. Bound states: identical prongs

We begin by discussing the interesting special case of N identical prongs, i.e. $V_i(x_i) = V(x_i)$. Note that this does not imply that the wavefunctions are identical on all the prongs. In fact, equation (3) implies that the wavefunctions are often different, since their derivatives at the vertex have to add up to zero. This statement is the generalization of the two-prong case, where one has symmetric and antisymmetric solutions.

The general bound-state solution for N identical prongs is easily obtained from the two-prong situation. For two identical prongs, one has the situation schematically shown in figure 2. We can define the single variable x such that $x = x_1$ for positive x, and $x = -x_2$ for negative x. Effectively, one is mapping the two prongs onto the real axis, $-\infty < x < \infty$. Clearly, this is the familiar situation of a symmetric one-dimensional potential. Its eigenstates correspond to even and odd solutions $\psi^{(n)}(x)$ at energies E_n, labelled by a quantum number n ($n = 0, 1, 2, \ldots$).

Theorem. The eigenstates of a potential with N identical prongs can be constructed from the eigenfunctions $\psi^{(n)}(x)$ of the corresponding symmetric two-prong system, and have exactly the same eigenenergies. Explicitly,

$$\psi^{(n)}(\vec{a}, \vec{x}) \equiv \left\{a_1\psi^{(n)}(x_1), a_2\psi^{(n)}(x_2), \ldots, a_N\psi^{(n)}(x_N)\right\} \qquad (x_i > 0)\,. \tag{6}$$

where $\vec{a}$ is a compact notation for the N-tuple $(a_1, a_2, \ldots, a_N)$. For even-numbered states the boundary conditions at the vertex imply $a_1 = a_2 = \cdots = a_N$; for odd-numbered states one has the constraint $\sum_i a_i = 0$, and there is an $(N-1)$-fold degeneracy.

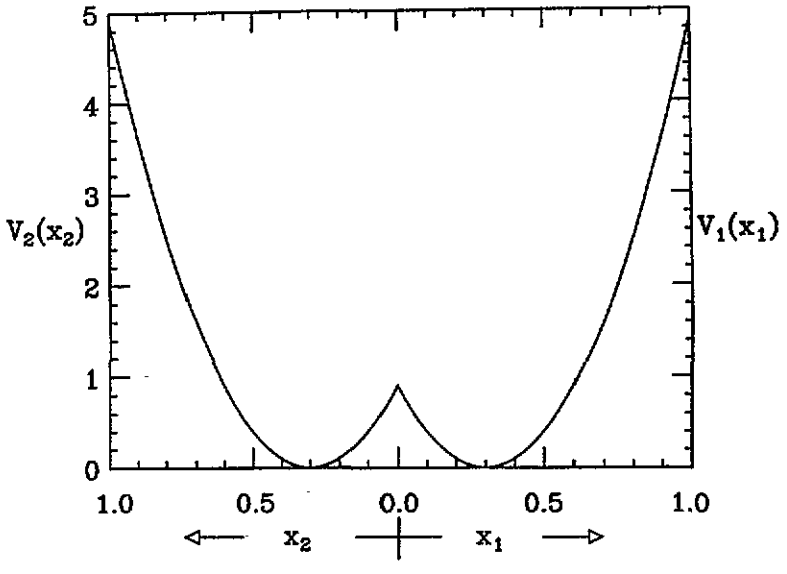

Figure 2. Example of a potential with two identical prongs. The positive variables x_1, x_2 together span the real x-axis ($-\infty < x < \infty$).

Proof. By construction, the wavefunction $\psi^{(n)}(\vec{a},\vec{x})$ for an eigenenergy E_n satisfies all the boundary conditions at the prong ends. It only remains to show that the vertex conditions, equations (2) and (3), are also satisfied by (6).

(i) For even eigenfunctions of a two-prong potential one has $\psi'(0)=0$, and in general $\psi(0)\neq 0$. Equation (3) is therefore trivially satisfied, and (2) implies $a_1=a_2=\cdots=a_N$. The eigenfunction is therefore determined up to one overall normalization constant, and the energy level is non-degenerate.

(ii) For odd eigenfunctions of a two-prong potential, one has $\psi'(0)\neq 0$, and $\psi(0)=0$. Now, equation (2) is trivially satisfied (since each wavefunction vanishes at the vertex) and the derivative condition (3) implies $\sum_i a_i=0$, as claimed. This constraint is not sufficient to fully determine the constants a_i, and the levels have an $(N-1)$-fold degeneracy. The completeness of states in (6) seems intuitively reasonable, and can be established from the completeness of the eigenfunctions of a symmetric one-dimensional potential.

As a first illustrative example, consider a particle of mass m moving in a harmonic potential with three identical prongs

$$V(x_i)=\tfrac{1}{2}m\omega^2x_i^2\,. \tag{7}$$

Each prong corresponds to the positive half of a harmonic oscillator potential. The ground state of this three-prong potential will have the same energy $E_0=\frac{1}{2}\hbar\omega$ as the two-prong case. The unnormalized eigenfunction is given by

$$\psi^{(0)}(\vec{a},\vec{x})\equiv\left\{a\psi^{(0)}(x_1),a\psi^{(0)}(x_2),a\psi^{(0)}(x_3)\right\} \tag{8}$$

where, $\psi^{(0)}\propto \mathrm{e}^{-\omega x^2/4}$ is the Gaussian harmonic oscillator ground-state wavefunction. We assume $\psi^{(n)}(x)$ to be normalized to unity over one prong, so that the wavefunction in (6) is properly normalized. The next level at energy $E_1=\frac{3}{2}\hbar\omega$ is doubly degenerate. Eigenfunctions at this level are given by

$$\psi^{(1)}(\vec{a},\vec{x})\equiv\left\{a_1\psi^{(1)}(x_1),a_2\psi^{(1)}(x_2),a_3\psi^{(1)}(x_3)\right\} \tag{9}$$

where $\psi^{(1)}\propto x\mathrm{e}^{-\omega x^2/4}$ is the wavefunction for the first excited state of the two-prong case. There is an infinite number of wavefunctions $\psi^{(1)}(\vec{a},\vec{x})$ that correspond to the same energy $\frac{3}{2}\hbar\omega$. Each of these wavefunctions is characterized by a set of coefficients a_i obeying the constraint $\sum_i a_i=0$. Due to this constraint, we see that all allowed a_i span a two-dimensional space, thus leading to a two-fold degeneracy at this level. The following two functions then provide one choice for an orthonormal basis in this two-dimensional two-fold degenerate space:

$$\begin{aligned}&\psi^{(1)}(\vec{a}_1;\vec{x}) \qquad \vec{a}_1\equiv\left(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}}\right)\\ &\psi^{(1)}(\vec{a}_2;\vec{x}) \qquad \vec{a}_2\equiv\left(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right).\end{aligned} \tag{10}$$

This pattern of non-degeneracy/degeneracy keeps repeating as we consider higher energy states. All even-numbered states are non-degenerate and all odd-numbered states have two-fold degeneracy. The three lowest eigenstates are shown in figure 3. The normalization has been chosen such that $\int_0^\infty \psi^{(n)*}(x)\psi^{(n)}(x)\,\mathrm{d}x=1$.

Similarly, the eigenfunctions for the potential $V(x_i)=0$, $x_i\leqslant 1$ are shown by the broken curves in figure 4. The wavefunctions are sinusoidal, and the energies are $E_n=n^2\pi^2/4$ $(n=1,2,\ldots)$, since they correspond to a symmetric two-prong potential which is an infinite square well of width 2.

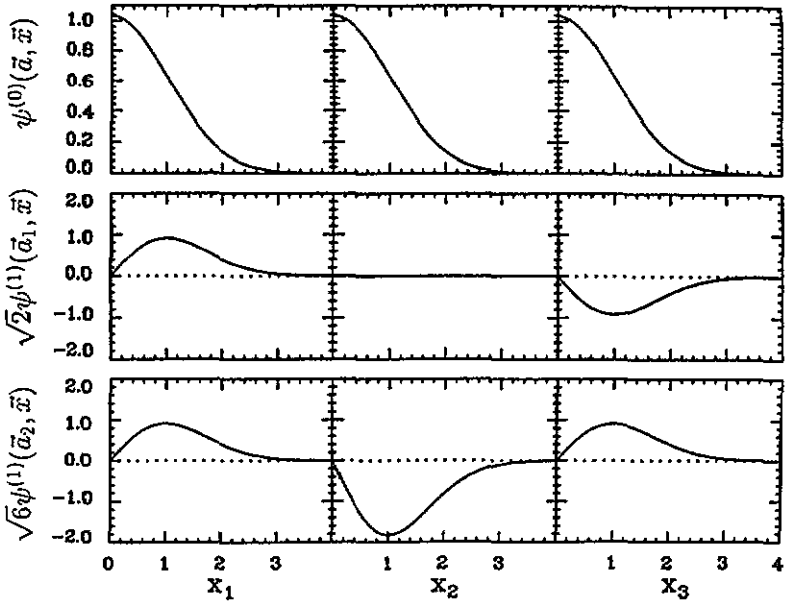

Figure 3. The three lowest eigenfunctions (equations (8)–(10)) for a three-prong harmonic oscillator potential $V(x_i) = \frac{1}{2}m\omega^2 x_i^2$. The ground state $\psi^{(0)}$ is non-degenerate, whereas the first excited state $\psi^{(1)}$ is doubly degenerate.

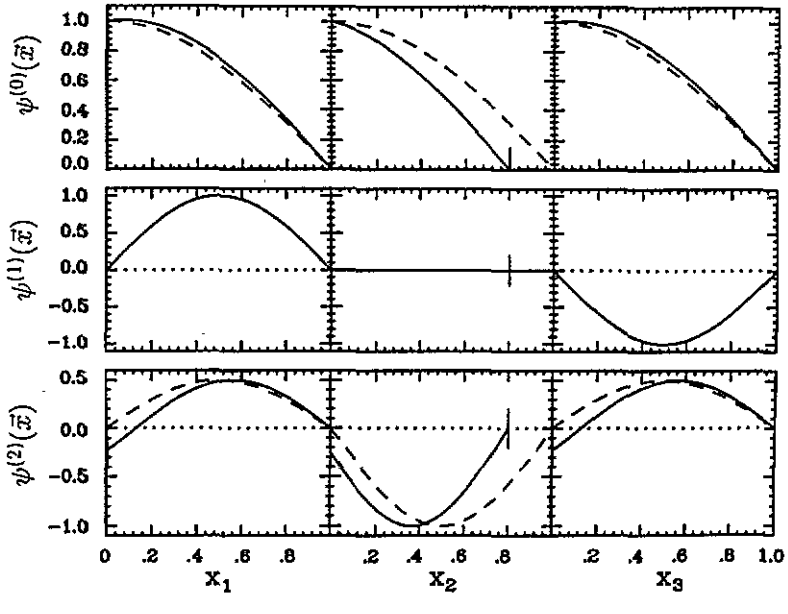

Figure 4. The three lowest energy eigenstates (unnormalized) for a three-prong free-particle potential. The full curves correspond to $l_1 = l_3 = 1$, $l_2 = 0.8$. The broken curves correspond to the identical prong case $l_1 = l_2 = l_3 = 1$.

From the above examples, it becomes clear that a potential with N identical prongs will have alternate non-degenerate and $(N-1)$-fold degenerate energy levels. Clearly, the

familiar non-degeneracy property of one-dimensional potentials is maintained for $N = 2$, but not for situations with more prongs.

4. Bound states: non-identical prongs

Let us now consider the general case of an N-prong system with potential $V_i(x_i)$ on the ith prong. Let $\psi_i(x_i)$ be the solution of the Schrödinger equation with energy E along that prong:

$$-\frac{d^2\psi_i}{dx_i^2} + (V_i - E)\,\psi_i = 0 \qquad (i = 1, 2, \ldots, N)\,.$$

Of the two linearly independent solutions to this equation, let $\phi_i(x_i, E)$ be the one that vanishes at the end point of the ith prong. This implies $\psi_i(x_i, E) = a_i\phi_i(x_i, E)$, where a_i is a constant. These $\psi_i(x_i, E)$, glued together properly so as to satisfy vertex requirements, will produce the wavefunction for the entire domain. From vertex conditions (2) and (3), we get

$$\psi_i(0, E) = a_i\phi_i(0, E) \equiv U \tag{11}$$

and

$$\sum_{i=1}^{N} \psi_i'(0, E) = \sum_{i=1}^{N} a_i\phi_i'(0, E) = 0\,. \tag{12}$$

The energy eigenvalues E are then determined by eliminating the unknowns a_i and U from the above set of $(N + 1)$ linear equations. The eigenvalue condition is

$$\det \begin{vmatrix} \phi_1(0,E) & 0 & \cdots & 0 & 1 \\ 0 & \phi_2(0,E) & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \phi_N(0,E) & 1 \\ \phi_1'(0,E) & \phi_2'(0,E) & \cdots & \phi_N'(0,E) & 0 \end{vmatrix} = 0\,. \tag{13}$$

For any given eigenvalue E_n, equation (11) then determines all the constants a_i in terms of one unknown constant U. The corresponding bound-state wavefunction is

$$\psi(\vec{x}, E_n) = U\left\{\frac{\phi_1(x_1, E_n)}{\phi_1(0, E_n)}, \ldots, \frac{\phi_N(x_N, E_n)}{\phi_N(0, E_n)}\right\}. \tag{14}$$

If desired, the constant U can also be fixed by requiring overall normalization of the wavefunction.

To illustrate the procedure for determining eigenvalues, consider the example of a particle that is free to move inside a domain of three prongs of lengths l_1, l_2 and l_3. Since the potential along the prongs is zero, the wavefunctions along them are given by sinusoidal functions that vanish at the end points. An eigenfunction of energy $E = k^2$ has the form

$$\psi(\vec{a}, \vec{x}) = \{a_1 \sin k(l_1 - x_1), a_2 \sin k(l_2 - x_2), a_3 \sin k(l_3 - x_3)\}\,. \tag{15}$$

In this case, equation (11) is

$$a_1 \sin kl_1 = a_2 \sin kl_2 = a_3 \sin kl_3 = U \tag{16}$$

where U is the common value of the wavefunction at the origin. The derivative condition (12) gives

$$a_1 \cos kl_1 + a_2 \cos kl_2 + a_3 \cos kl_3 = 0\,. \tag{17}$$

The eigenvalue condition (13), when simplified, reads

$$\cos kl_1 \sin kl_2 \sin kl_3 + \sin kl_1 \cos kl_2 \sin kl_3 + \sin kl_1 \sin kl_2 \cos kl_3 = 0. \tag{18}$$

As a specific case, consider the situation $l_2 = 0.8$ and $l_1 = l_3 = 1$. For this case, the solutions to (18) are found to be $k = 1.68, \pi, 3.61, 5.08, 2\pi, 7.17, 8.54, 3\pi, \ldots$. The solutions $k = m\pi$ $(m = 1, 2, \ldots)$ are a consequence of maintaining partial symmetry by taking two prongs to be identical ($l_1 = l_3 = 1$). The corresponding wavefunctions are shown by the full curves in figure 4. (For comparison, we have also plotted the broken curves corresponding to the wavefunctions for the case of all three identical prongs of length 1.) From the figure, it is apparent that the ground-state wavefunction at $E_0 = 1.68^2$ has no nodes. The first excited state at $E_1 = \pi^2$ has one node at the vertex point. Note that for this state, the wavefunction in prong 2 is zero. The second excited state is at energy $E_2 = 3.61^2$ and has two nodes, one in prong 1 and the other in prong 3. This result is very suggestive—one expects one extra node to appear for each higher eigenstate, similar to the familiar one-dimensional situation.

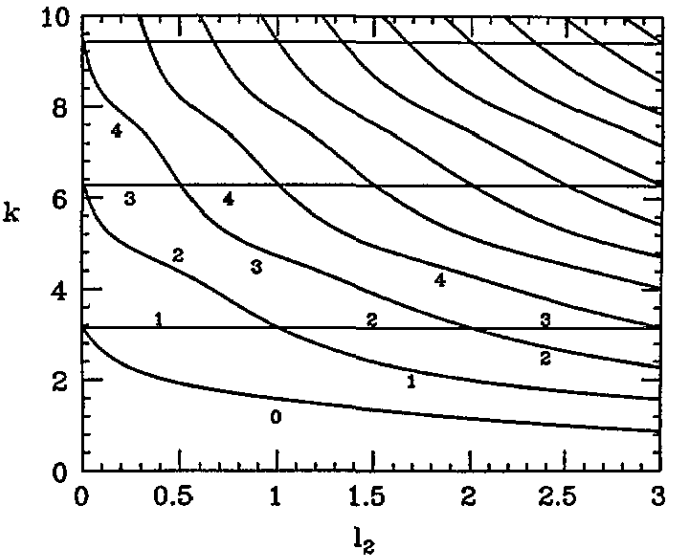

Figure 5. The eigenvalues k (energy $E = k^2$) for a three-prong free particle potential as a function of l_2, the length of prong 2. The lengths of the other two prongs are kept fixed and *equal* $l_1 = l_3 = 1$. Note the pattern of degeneracy and level crossings. The curves are labelled by the number of nodes in the wavefunction (in all prongs).

We have also studied the variation of the eigenvalues k systematically as the prong length l_2 is varied. The results are shown in figure 5. As discussed above, the solutions $k = m\pi$ are present, and when l_2 is an integral multiple of $l_1 (= l_3)$, one has the interesting occurrence of degeneracy and level crossing. The curves in figure 5 are labelled by the number of nodes in the wavefunctions. At any fixed value of l_2, the number of nodes increases with energy. Note that as $l_2 \to 0$, the eigenvalues become doubly degenerate at $k = m\pi$, $(m = 1, 2, \ldots)$. This is intuitively clear since the limit $l_2 \to 0$ forces the wavefunction to vanish at the vertex, thereby effectively breaking the problem into two infinite square-well potentials of widths $l_1 = 1$ and $l_3 = 1$. Finally, in figure 6, we take $l_1 \neq l_3$, and plot eigenvalues k. This is a completely asymmetric situation, and there is

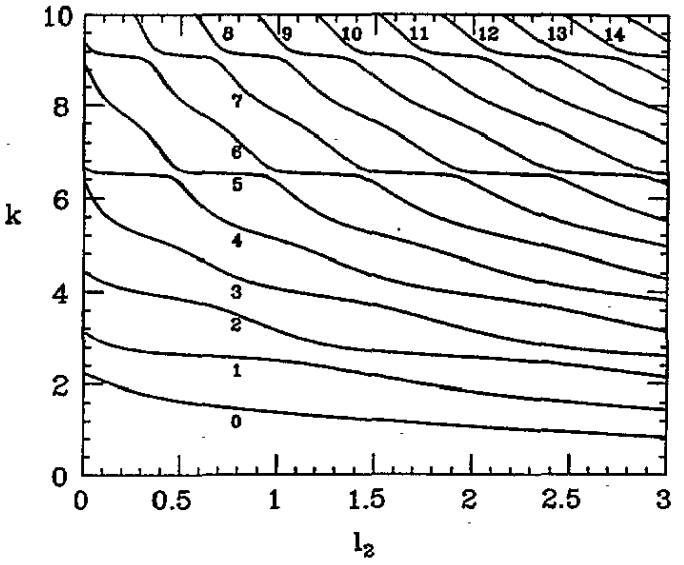

Figure 6. The eigenvalues k (energy $E = k^2$) for a three-prong free particle potential as a function of l_2, the length of prong 2. The lengths of the other two prongs are kept fixed and *unequal* $l_1 = 1$, $l_3 = \sqrt{2}$. There is now no degeneracy. The curves are labelled by the number of nodes in the wavefunction (in all prongs).

now no degeneracy of energy levels. Again note that the number of nodes in the overall wavefunction increases by one with increasing energy levels.

As a second example, we determine the eigenstates of a three-prong potential $V_i(x_i) = \frac{1}{4}\omega_i^2 x_i^2$, composed of three harmonic oscillators of different angular frequencies $\omega_i (i = 1, 2, 3)$, and $\hbar = 2m = 1$. The wavefunctions which vanish at $x_i \to \infty$ are [24]

$$\psi_i(x_i) = a_i D_{\nu_i}(\sqrt{\omega_i} x_i) \qquad \nu_i = \frac{E}{\omega_i} - \frac{1}{2} \tag{19}$$

where D_ν is a parabolic cylinder function. The eigenvalue condition (equation (13)) gives

$$\sqrt{\omega_1} D'_{\nu_1}(0) D_{\nu_2}(0) D_{\nu_3}(0) + \text{cyclic permutations} = 0. \tag{20}$$

The parabolic cylinder function and its derivative at the origin have simple expressions in terms of gamma functions [24]:

$$D_\nu(0) = 2^{\frac{\nu}{2}} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{\nu}{2})} \qquad D'_\nu(0) = 2^{\frac{\nu}{2} - \frac{1}{2}} \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})}. \tag{21}$$

The solutions of the eigenvalue equation are then easily found. For the choice $\omega_1 = 1.0$, $\omega_2 = 2.0$, $\omega_3 = 3.0$, one gets the four lowest eigenstates at energies $E = 0.83, 1, 94, 3.29, 3.85$. There is no degeneracy since all the ω are different.

In the above two examples, exact analytic forms for the solutions $\psi_i(x_i)$ were available. Even if this is not the case, it is easy to use numerical Runge–Kutta techniques applied to each prong.

5. Supersymmetric quantum mechanics on multi-pronged domains

Given any one-dimensional potential $V(x)$, the powerful techniques of supersymmetric quantum mechanics can be used to generate a partner potential $\tilde{V}(x)$ with the same

eigenvalues [17, 18]. This property has been extensively used to get a deeper understanding of exactly solvable potentials and for generating improved approximation methods (SWKB method [25], large N-method [26], etc) for determining eigenvalues. Supersymmetric quantum mechanics provides an elegant formalism which includes and goes substantially beyond the method of factorization previously applied to some potentials [27]. In this section, we show how the ideas of supersymmetric quantum mechanics can be applied to a given N-prong system $V(\vec{x})$ in order to generate solutions for a new N-prong potential $\tilde{V}(\vec{x})$.

As in the previous section, consider a scalar N-prong potential $V(\vec{x})$ composed of potentials $V_i(x_i)$ on prong i $(i = 1, 2, \ldots, N)$. Its eigenvalues and eigenfunctions are then given by (13) and (14), respectively. The unnormalized ground-state wavefunction is

$$\psi^{(0)}(\vec{x}, E_0) \propto \left\{ \frac{\phi_1^{(0)}(x_1, E_0)}{\phi_1^{(0)}(0, E_0)}, \ldots, \frac{\phi_N^{(0)}(x_N, E_0)}{\phi_N^{(0)}(0, E_0)} \right\} . \tag{22}$$

This can be used to define the superpotential $W(\vec{x})$ whose value on prong i is

$$W_i(x_i) = -\frac{\phi_i^{(0)'}(x_i)}{\phi_i^{(0)}(x_i)} . \tag{23}$$

It is easy to check that

$$V_i(x_i) = W_i^2(x_i) - W_i'(x_i) + E_0 . \tag{24}$$

The supersymmetric partner potential is given by

$$\tilde{V}_i(x_i) = W_i^2(x_i) + W_i'(x_i) + E_0 . \tag{25}$$

The potentials $\tilde{V}_i(x_i)$ taken together make up the full N-prong potential $\tilde{V}(\vec{x})$. From supersymmetric quantum mechanics, we know that the solution of the Schrödinger equation for potential $\tilde{V}_i(x_i)$ and energy E is given by [18]

$$\tilde{\phi}_i(x_i, E) = a_i \left(-\frac{\mathrm{d}}{\mathrm{d}x_i} + \frac{\phi_i^{(0)'}(x_i)}{\phi_i^{(0)}(x_i)} \right) \phi_i(x_i, E) \tag{26}$$

where a_i are constants. Since $\phi_i(x_i, E)$ vanishes at the prong ends, so do all the $\tilde{\phi}_i(x_i, E)$. At the vertex, one wants

$$\tilde{\phi}_i(0, E) = \tilde{\phi}_j(0, E) \qquad \text{and} \qquad \sum_i \tilde{\phi}_i'(0, E) = 0 . \tag{27}$$

This gives an eigenvalue condition similar to (13):

$$\det \begin{vmatrix} \tilde{\phi}_1(0,E) & 0 & \cdots & 0 & 1 \\ 0 & \tilde{\phi}_2(0,E) & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \tilde{\phi}_N(0,E) & 1 \\ \tilde{\phi}_1'(0,E) & \tilde{\phi}_2'(0,E) & \cdots & \tilde{\phi}_N'(0,E) & 0 \end{vmatrix} = 0 . \tag{28}$$

In general, the eigenvalues obtained from (28) will be different from those coming from (13). However, there are two important situations where the *same* eigenvalues result. This happens for the special case of two prongs ($N = 2$) and for the case of all identical prongs. These results can be understood from a physical viewpoint, since $N = 2$ is the standard one-dimensional situation treated in supersymmetric quantum mechanics, and furthermore, as we have seen in section 3, these are the same eigenvalues for the identical-prong case. Thus we see the machinery of supersymmetric quantum mechanics can be immediately applied to get

eigenvalues and eigenfunctions of the partner potential $\tilde{V}(\vec{x})$ for the identical-prong case. The equality of eigenvalues from (13) and (28) can also be established from a mathematical viewpoint using (26) and (27).

6. Perturbation theory

Having solved a multi-prong problem, it is of interest to see the influence of a small perturbation of the potential in one of the prongs, say, on the energy levels. Here, for simplicity, we will only deal with an unperturbed system with three identical prongs. Consider an unperturbed system with potential $V_i(x_i) = V_0(x_i)$ on each prong. Its completely symmetric ground state

$$\psi^{(0)}(\vec{1}, \vec{x}) \equiv \frac{1}{\sqrt{3}}\{\psi^{(0)}(x_1), \psi^{(0)}(x_2), \psi^{(0)}(x_3)\} \tag{29}$$

is described in terms of $\psi^{(0)}(x)$; the ground-state eigenfunction of a two-prong system with the same potential $V_0(x_i)$ in each prong. We have chosen $\psi^{(0)}(x)$ to be normalized to unity over one prong, so that the wavefunction in (29) is properly normalized. The eigenenergy of this state is also the same as the ground-state energy of the two-prong system. The first excited state for this three-prong system is doubly degenerate. We choose the following two orthonormal states as our base states:

$$\psi^{(1)}(\vec{a}_1; \vec{x}) \equiv \frac{1}{\sqrt{2}}\{\psi^{(1)}(x_1), 0, -\psi^{(1)}(x_3)\} \tag{30}$$

and

$$\psi^{(1)}(\vec{a}_2; \vec{x}) \equiv \frac{1}{\sqrt{6}}\{\psi^{(1)}(x_1), -2\psi^{(1)}(x_2), \psi^{(1)}(x_3)\}\,. \tag{31}$$

Both of these states have exactly the same energy as the first excited state of the two-prong system.

Now, let us include a perturbation V_I that is non-vanishing in only one prong. The energy of the system is shifted and the degeneracy of the states of (30) and (31) is lifted. Interestingly, only one of the eigenvalues of the first excited state changes, while the other one remains the same, thus causing the split. We compute here such shifts for the ground state as well the first excited state of the system. For numerical concreteness only, in the following example, we use the harmonic oscillator potential for V_0 and αx_1^3 as the perturbation potential V_I on prong 1. Thus the unperturbed potential is

$$V_0(x_i) = \tfrac{1}{4}\omega^2 x_i^2\,. \tag{32}$$

Following the usual route, the shift of the ground state can be computed. The first-order shift, δE_0, is given by

$$\delta E_0 = \int \mathrm{d}x_1\, \psi_1^{(0)*}(x_1) V_I \psi_1^{(0)}(x_1) = \frac{4\alpha}{3\sqrt{2\pi\omega^3}}\,. \tag{33}$$

The next energy level has two-fold degeneracy, and hence it is necessary to use the formalism of degenerate perturbation theory. We compute the matrix elements of the perturbing potential V_I using $\psi^{(1)}(\vec{a}_1; \vec{x})$ and $\psi^{(1)}(\vec{a}_2; \vec{x})$ as basis vectors. For this situation, the matrix V_I is explicitly given by

$$V_I = \begin{pmatrix} \gamma & \frac{1}{\sqrt{3}}\gamma \\ \frac{1}{\sqrt{3}}\gamma & \frac{1}{3}\gamma \end{pmatrix}$$

where

$$\gamma = \frac{8\alpha}{\sqrt{2\pi\omega^3}}.$$

Diagonalizing the corresponding Hamiltonian, we find the eigenvalues to be 0 and $\frac{4}{3}\gamma$. Hence, at this level, degeneracy is lifted and the difference in energy is $\frac{4}{3}\gamma$. The two eigenfunctions are given by

$$\frac{1}{2}\psi^{(1)}(\vec{a}_1;\vec{x}) - \frac{\sqrt{3}}{2}\psi^{(1)}(\vec{a}_2;\vec{x}) = \frac{1}{\sqrt{2}}\{0, \psi^{(1)}(x_2), -\psi^{(1)}(x_3)\} \tag{34}$$

and

$$\frac{1}{2}\psi^{(1)}(\vec{a}_1;\vec{x}) + \frac{\sqrt{3}}{2}\psi^{(1)}(\vec{a}_2;\vec{x}) = \frac{1}{\sqrt{6}}\{2\psi^{(1)}(x_1), -\psi^{(1)}(x_2), \psi^{(1)}(x_3)\}. \tag{35}$$

It is interesting to note that the first state has the same energy as the unperturbed system. Thus we see that, since the perturbation was added to just one prong, it does not break the symmetry completely and one of the eigenenergies remains unchanged.

7. Scattering in a multi-prong system

The scattering of an incident plane wave off the vertex in a multi-prong system offers scenarios that are different from the usual scattering in one-dimensional problems. As a first example, consider a plane wave with energy E moving along prong 1, incident upon the vertex of an N-prong system with a constant potential $V = 0$ on all prongs. This is a trivial example in the two-prong case leading to 100% transmission, and zero reflection. For more than two prongs, it is not *a priori* intuitively clear whether full transmission will occur or not. The wavefunction on prong 1 is given by

$$\psi_1(x_1) = \exp(-ikx_1) + r\exp(ikx_1) \tag{36}$$

where k is the momentum of the incident plane wave ($E = k^2$). The wavefunction on all the other prongs ($i \neq 1$) only consists of outgoing waves and is given by

$$\psi_i(x_i) = t_i \exp(ikx_i). \tag{37}$$

Imposition of the boundary conditions (2) and (3) on these wavefunctions at the vertex relates the reflected amplitude r to the transmission amplitudes t_i. Their explicit relationship is given by

$$(1+r) = t_i \qquad (i = 2, \ldots, N) \qquad r - 1 + \sum_{i=2}^{N} t_i = 0. \tag{38}$$

The solution is

$$r = \frac{2-N}{N} \qquad t_i = \frac{2}{N} \qquad (i = 2, \ldots, N). \tag{39}$$

The reflection and transmission coefficients are then given by

$$\mathcal{R} = \frac{(2-N)^2}{N^2} \qquad \mathcal{T}_i = \frac{4}{N^2} \qquad (i = 2, \ldots, N).$$

Clearly, one has $\mathcal{R} + \sum_{i=2}^{N} \mathcal{T}_i = 1$, and the probability current is conserved. For $N = 2$, the reflection coefficient vanishes, as expected. However, for $N > 2$ there is always a finite amount of reflection. For example, for a three-prong potential, the reflection coefficient is

$\mathcal{R} = \frac{1}{9}$. In fact $\mathcal{R}$ increases continuously with N, and approaches unity for a large number of prongs, N. (This increase of $\mathcal{R}$ with N is shown in figure 7 by the curve labelled 1.) Thus one has the curious result that an incident wave, when given a large choice of scattering paths, in fact prefers to be reflected back on its initial prong!

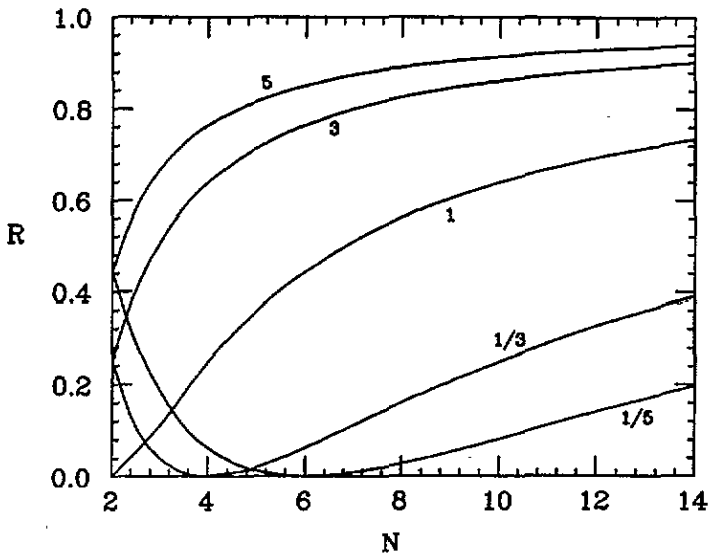

Figure 7. A plot of the reflection coefficient $\mathcal{R}$ versus the number of prongs N. The incident wave with energy E is on prong 1 with zero potential. All other prongs are taken to be at a constant potential V_0. The curves are labelled by the parameter $\zeta \equiv \sqrt{1 - V_0/E}$.

Now let us consider a more general case where the incident wave is on a prong with potential zero, whereas the remaining $(N-1)$ prongs are at constant potential V_0. Let the energy of the wave be $E > V_0$. Let us define $k^2 = E$, $k'^2 = E - V_0$ and the parameter

$$\zeta = \frac{k'}{k} = \sqrt{1 - \frac{V_0}{E}}\,.$$

The wavefunction on prong 1 is again given by $\psi_1(x_1) = \exp(-ikx_1) + r\exp(ikx_1)$, and the wavefunctions along the remaining $(N-1)$ prongs are given by $\psi_i(x_i) = t_i \exp(ik'x_i)$ $(i = 2, \ldots, N)$. Boundary conditions (2) and (3) applied at the vertex yield

$$(1+r) = t_2 = t_3 = \cdots = t_N \equiv t \qquad -1 + r + (N-1)\zeta = 0\,. \tag{40}$$

Solving these equations, we find

$$r = \frac{1-\zeta(N-1)}{1+\zeta(N-1)} \qquad \text{and} \qquad t = \frac{2}{1+\zeta(N-1)}\,.$$

One can show that r and t obey $1 = r^2 + \zeta(N-1)t^2$, as required by conservation of probability. The reflection coefficient $\mathcal{R}$, which is now given by

$$\mathcal{R} = \left[\frac{1-\zeta(N-1)}{1+\zeta(N-1)}\right]^2$$

has a rather interesting behavior as a function of ζ and N. In figure 7, we have plotted $\mathcal{R}$ as a function of N for several values of ζ. $\mathcal{R}$ vanishes for $\zeta = 1/(N-1)$. Thus for some

special values of V_0/E, we have complete transmission through the vertex. As N increases, $\mathcal{R}$ slowly approaches unity for all ζ.

As yet another interesting example we consider scattering from a T-stub with N open ended prongs and one prong of finite length l (the stub). We consider a case where potentials along all the prongs are zero. A plane wave of momentum k (energy $E = k^2$) incident upon the vertex from prong 1, sets up stationary waves in the stub, and outgoing plane waves in the remaining $N-1$ prongs. The wavefunction on prong 1 is given by $\exp(-ikx_1)+r\exp(ikx_1)$. Along the stub, the wavefunction is given by $A\sin k(l-x_s)$, where x_s is the coordinate along the stub and A is a constant. Wavefunctions along all other prongs are given by $t\exp(ikx_j)$, $(j = 2, \ldots, N)$. From boundary conditions (2) and (3), we get

$$1 + r = A\sin kl = t \qquad ik(r-1) + ik(N-1)t - Ak\cos kl = 0\,. \tag{41}$$

Solving them, we get

$$r = \frac{-(N-2) - \cot kl}{N + i\cot kl} \qquad t = \frac{2}{N + i\cot kl}\,.$$

The reflection coefficient is given by

$$\mathcal{R} \equiv |r|^2 = \frac{\cot^2 kl + (N-2)^2}{\cot^2 kl + N^2}\,.$$

Clearly, from the above expression, one gets $|r|^2 + (N-1)|t|^2 = 1$. Interestingly, we find that for $kl = n\pi$, i.e. energy $E = n^2\pi^2/l^2$, the reflection coefficient $\mathcal{R} \to 1$. At these energies, a standing wave is set up jointly in the stub and in prong 1, and the particle never enters the other prongs.

Let us generalize the discussion of scattering to an N-prong vertex with identical potentials $V(x_i)$ on all prongs. Define two linearly independent solutions $f(x_i)$ and $g(x_i)$ of the Schrödinger equation using asymptotic boundary conditions as $x_i \to \infty$:

$$f(x_i) \to \exp(-ikx_i) \qquad g(x_i) \to \exp(ikx_i)\,. \tag{42}$$

The vertex conditions are $f(0) + r\,g(0) = t\,g(0)$, and $f'(0) + r\,g'(0) + (N-1)\,t\,g'(0) = 0$. The solution is

$$r = -\frac{1}{N}\frac{f'(0)}{g'(0)} - \frac{(N-1)}{N}\frac{f(0)}{g(0)} \qquad t = -\frac{1}{N}\frac{f'(0)}{g'(0)} + \frac{1}{N}\frac{f(0)}{g(0)}\,. \tag{43}$$

Conservation of probability requires $|r|^2 + (N-1)|t|^2 = 1$ for all values of N. This is satisfied provided

$$\left|\frac{f(0)}{g(0)}\right| = \left|\frac{f'(0)}{g'(0)}\right| = 1\,.$$

Here again, we see that as the number of prongs $N \to \infty$, the reflection coefficient $\mathcal{R} = |r|^2$ approaches unity.

Having analysed scattering on a general identical N-prong domain, we now investigate whether we can extract any further information using supersymmetry. In one-dimensional quantum mechanics, supersymmetry relates reflection and transmission coefficients of one potential with those of its supersymmetric partner potential [23]. We find that this relationship holds also for the case of N identical prongs.

We have just seen that for the potential $V(x_i)$ on all prongs, the scattering is described by (43). The partner potential is given by (25). Let $\tilde{f}$ and $\tilde{g}$ be the solutions of the

partner potential which behave like $\tilde{f}(x_i) \to \exp(-ikx_i)$, $\tilde{g}(x_i) \to \exp(ikx_i)$, as $x_i \to \infty$. Functions $\tilde{f}$ and $\tilde{g}$ are given by

$$\tilde{f}(x_i) = \frac{1}{W_\infty + ik}\left\{-\frac{d}{dx_i} + W(x_i)\right\} f(x_i) \qquad \tilde{g}(x_i) = \frac{1}{W_\infty - ik}\left\{-\frac{d}{dx_i} + W(x_i)\right\} g(x_i) \tag{44}$$

where W_∞ is the value of the superpotential at infinity. From these expressions, we can show that

$$\frac{\tilde{f}(0)}{\tilde{g}(0)} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\frac{f'(0)}{g'(0)} \qquad \text{and} \qquad \frac{\tilde{f}'(0)}{\tilde{g}'(0)} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\frac{f(0)}{g(0)}.$$

The reflection and transmission amplitudes for the potential $\tilde{V}(x_i)$ are given by

$$\tilde{r} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\left(-\frac{1}{N}\frac{f(0)}{g(0)} - \frac{(N-1)}{N}\frac{f'(0)}{g'(0)}\right)$$
$$\tilde{t} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right)\left(-\frac{1}{N}\frac{f(0)}{g(0)} + \frac{1}{N}\frac{f'(0)}{g'(0)}\right).$$

Note, for the special case of two-prongs, these equations agree with [23]. We find the following relationship among r, t and their partners $\tilde{r}$ and $\tilde{t}$:

$$\tilde{r} = \left(\frac{W_\infty - ik}{W_\infty + ik}\right) r \qquad \tilde{t} = -\left(\frac{W_\infty - ik}{W_\infty + ik}\right) t.$$

Note that $|\tilde{r}|^2 = |r|^2$, which implies that for the identical-prong case, the reflection coefficients are the same for any potential and its supersymmetric partner.

Now, using an explicit example, we provide a concrete demonstration of the usefulness of the machinery that we just developed. For the sake of simplicity, we shall work with the example of a free particle in an N-prong system for which reflection and transmission coefficients are given by (39). The superpotential for this problem is given by $W(x) = \tanh x$. This superpotential generates two distinct potentials related by supersymmetry. They are $V(x) \equiv W^2(x) - W'(x) = 1 - 2\,\mathrm{sech}^2 x$ and $\tilde{V}(x) \equiv W^2(x) + W'(x) = 1$. The first potential holds one bound state at energy $E = 0$.

Amplitudes $\tilde{r}$ and $\tilde{t}$ for the free particle system are $(2-N)/N$ and $2/N$, respectively. Amplitudes r and t for the potential $V(x) = 1 - 2\,\mathrm{sech}^2 x$, are then given by

$$r = \left(\frac{1+ik}{1-ik}\right)\left[-\frac{1}{N}\frac{\tilde{f}(0)}{\tilde{g}(0)} - \frac{N-1}{N}\frac{\tilde{f}'(0)}{\tilde{g}'(0)}\right] = \left(\frac{1+ik}{1-ik}\right)\left(\frac{N-2}{N}\right)$$
$$t = -\left(\frac{1+ik}{1-ik}\right)\left(\frac{2}{N}\right).$$

Thus, knowing the reflection and transmission amplitudes $\tilde{r}$ and $\tilde{t}$ for the free-particle system, supersymmetry allows us to determine amplitudes r and t for the rather non-trivial potential $V(x) = 1 - 2\,\mathrm{sech}^2 x$. Note that coefficients $\tilde{\mathcal{R}} = |\tilde{r}|^2$ and $\tilde{\mathcal{T}} = |\tilde{t}|^2$ are equal to coefficients for the partner potential, $\mathcal{R} = |r|^2$ and $\mathcal{T} = |t|^2$, respectively.

8. Tunnelling in a multi-prong system

Tunnelling is another sector of great interest that shows a marked difference from the two-prong case. Consider a system with N identical prongs, each one having one minimum (well-like structure, similar to the two-prong case shown in figure 2). In this section, we

look at the tunnelling of a localized wavepacket from one prong to the others. For simplicity, we start with a three-prong system. The generalization to higher number of prongs is then straightforward. Let $\Phi_i(\vec{x})$ denote a wavepacket localized in the ith prong. Such a packet can be approximated by the following linear combination of eigenstates of the system:

$$\Phi_i(\vec{x}) = \tfrac{1}{3}\left(\psi^{(0)}(\vec{1},\vec{x}) + \psi^{(1)}(\vec{a}_i,\vec{x})\right) \tag{45}$$

where $\vec{1} \equiv (1,1,1)$ and $\vec{a}_i$ denotes a three-tuple that has a 2 in the ith position and -1 in the rest. For example, $\vec{a}_1$ is given by the three-tuple $(2,-1,-1)$.

The amplitude for the localized state $\Phi_i(\vec{x})$ to tunnel out of the prong 1 is given by

$$\mathcal{A}(t) = \langle\psi_f|\exp(-\mathrm{i}Ht)|\psi_i\rangle \tag{46}$$

where

$$\psi_i = \Phi_1(\vec{x}) \qquad \text{and} \qquad \psi_f = \frac{1}{\sqrt{\alpha_2^2+\alpha_3^2}}\left(\alpha_2\Phi_2(\vec{x}) + \alpha_3\Phi_3(\vec{x})\right)$$

$(\alpha_2+\alpha_3=1)$. Substituting explicit expressions for ψ_f and ψ_i in (46), we get

$$\begin{aligned}\mathcal{A}_{\alpha_2\alpha_3} &= \frac{1}{3\sqrt{\alpha_2^2+\alpha_3^2}}\int \mathrm{d}\vec{x}\left[\alpha_2\Phi_2(\vec{x})+\alpha_3\Phi_3(\vec{x})\right]\left[\mathrm{e}^{-\mathrm{i}E_0t}\psi^{(0)}(\vec{1},\vec{x}) + \mathrm{e}^{-\mathrm{i}E_1t}\psi^{(1)}(\vec{a}_1,\vec{x})\right] \\ &= \frac{1}{9\sqrt{\alpha_2^2+\alpha_3^2}}\int \mathrm{d}\vec{x}\left[\alpha_2\psi^{(1)}(\vec{a}_2,\vec{x}) + \alpha_3\psi^{(1)}(\vec{a}_3,\vec{x}) + \psi^{(0)}(\vec{1},\vec{x})\right] \\ &\quad\times\left[\mathrm{e}^{-\mathrm{i}E_0t}\psi^{(0)}(\vec{1},\vec{x}) + \mathrm{e}^{-\mathrm{i}E_1t}\psi^{(1)}(\vec{a}_1,\vec{x})\right] \\ &= \frac{\mathrm{e}^{-\mathrm{i}E_1t}}{9\sqrt{\alpha_2^2+\alpha_3^2}}\left[\alpha_2\vec{a}_1\cdot\vec{a}_2 + \alpha_3\vec{a}_1\cdot\vec{a}_3 + 3\mathrm{e}^{\mathrm{i}\delta Et}\right] \qquad \delta E \equiv E_1 - E_0\end{aligned}$$

where we have used $\int \mathrm{d}\vec{x}\,\psi^{(1)}(\vec{a}_1,\vec{x})\psi^{(1)}(\vec{a}_2,\vec{x}) = \vec{a}_1\cdot\vec{a}_2 = \sum_i(\vec{a}_1)_i(\vec{a}_2)_i$. Substituting $\vec{a}_1\cdot\vec{a}_3 = \vec{a}_1\cdot\vec{a}_2 = -3$, we get

$$\mathcal{A}_{\alpha_2\alpha_3} = \frac{\mathrm{e}^{-\mathrm{i}E_1t}}{3\sqrt{\alpha_2^2+\alpha_3^2}}\left[-1+\mathrm{e}^{\mathrm{i}\delta Et}\right].$$

Therefore the probability of tunnelling out of prong 1 into prong 2 or prong 3 is

$$\mathcal{P}_{\alpha_2\alpha_3} = \left|\mathcal{A}_{\alpha_2\alpha_3}\right|^2 = \frac{4}{9(\alpha_2^2+\alpha_3^2)}\sin^2\left(\frac{\delta Et}{2}\right).$$

The maximum value for this tunnelling probability is $\frac{8}{9}$, and that occurs when $\alpha_2=\alpha_3=\frac{1}{2}$. It is worth noting here that, unlike the two-prong case, the wavepacket never completely goes out of the prong 1.

When we generalize this situation to the N-prong case, we find that the probability amplitude for a wavepacket $\Phi_1(\vec{x})$ of tunnelling out of prong 1 is given by

$$\mathcal{A}_{\vec{\alpha}} = \frac{2\mathrm{i}\mathrm{e}^{-\mathrm{i}(E_0+\delta E/2)}\sin(\delta Et/2)}{N\left(\sum_{j=2}^N\alpha_j^2\right)^{1/2}}$$

where $\vec{\alpha} \equiv \{\alpha_2,\alpha_3,\ldots,\alpha_N\}$ with $\alpha_i/\left(\sum_{j=2}^N\alpha_j^2\right)$ giving the amplitude for finding the packet in the ith prong as

$$\left(\sum_i \alpha_i\Phi_i(\vec{x})\left(\sum_{j=2}^N\alpha_j^2\right)\right).$$

This amplitude is maximum for the symmetric case for which all α_i are equal. In this case the probability of tunnelling is given by

$$\mathcal{P} = \frac{4(N-1)}{N^2} \sin^2\left(\frac{\delta E\, t}{2}\right).$$

Thus the probability of tunnelling decreases when more alternatives are available, and goes to zero as the number of prongs becomes very large. This very surprising result is conceptually similar to the scattering situation discussed in section 7.

Acknowledgments

It is a real pleasure for one of us (US) to thank Dr Arun Jayannavar for helpful initial discussions. This work was supported in part by the US Department of Energy under grant DE-FGO2-84ER40173. We would also like to thank the two referees for many suggestions, and bringing to our notice a few relevant references. In particular, we would like to mention that the authors of [12] have studied the case of $N = 4$ in greater detail.

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